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Competing Condensates in Two Dimensions

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Abstract

We generalize our previous 2-dimensional model in which a pairing condensate $\langle\psi\psi\rangle$ was generated at large N. In the present case, we allow for both $\langle\psi\psi\rangle$ and a chiral condensate $\langle\bar{\psi}\psi\rangle$ to exist. We construct the effective potential to leading order in $\frac{1}{N}$, and derive the gap equations at finite density and temperature. We study the zero density and temperature situation analytically. We perform the renormalization explicitly and we show that the physics is controlled by a parameter related to the relative strengths of the interactions in the pairing and chiral channels. We show that although a solution to the gap equations exists in which both condensates are non-vanishing, the global minimum of the effective potential always occurs for the case when one or the other condensate vanishes.

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I. INTRODUCTION

In a previous paper [1], we introduced a variant of the Gross-Neveu model which, in the large N limit, exhibits the formation of a pairing condensate $\langle\psi\psi\rangle$. In that work, we derived and solved the gap equation, demonstrated that the coupling was asymptotically free, and discussed some of the properties of the relevant Green's functions. Our work was motivated, at least in part, by recent interest in the formation of similar condensates in QCD in the presence of a chemical potential. [2-4]

But in real QCD, as well as in a variety of condensed-matter systems, the pairing condensate is not the only possible one. In QCD at low density and temperature there is a chiral condensate; it is only as the density increases that one believes the $\langle\bar{q}q\rangle$ condensate disappears and a new phase, characterized by a non-vanishing $\langle qq\rangle$, replaces it. Likewise, in other systems, which condensates exist depends on external parameters like the density and temperature, and also on the relative strengths of various couplings. With this in mind, we extend our previous work by considering a more general model governed by two independent couplings: the original Gross-Neveu term [5] that promotes the condensation of $\langle\bar{\psi}\psi\rangle$, and the term considered in our earlier paper that produces a $\langle\psi\psi\rangle$ condensate. As in ref. 1, we shall be able to write down the gap equations exactly to leading order in $\frac{1}{N}$, and to renormalize the couplings, thereby rendering the gap equations finite. For the general case of non-vanishing density and temperature, we shall be able to write the gap equations in terms of a single integral over a momentum variable k; the effective potential is then expressible as a further integral over the auxiliary fields. To solve the gap equations will require numerical evaluation of these integrals, which takes us beyond the scope of the present work.

If we set temperature and chemical potential to zero, we can derive a closed-form expression for the effective potential, and can explicitly perform the renormalization of the couplings. We then find that there is one dimensionless parameter, independent of the renormalization scale, whose value determines which of the condensates is present. This situation might be described as "partial dimensional transmutation": the unrenormalized theory has two bare couplings whereas the renormalized one has a renormalization scale, which is arbitrary, and a dimensionless parameter, independent of this scale that controls the physics. We find that the gap equations have three types of solution: two in which one or the other of the condensates vanish, and a third, mixed case, in which both condensates are non-vanishing. It turns out, however, that the true minimum of the effective potential is always at a point where one of the condensates vanishes. Thus the mixed case is never realized physically, at least for zero temperature and density.

In the remainder of the paper, we shall describe the analysis that leads to the above results. In section II, we define the model and derive the gap equations for the general case of non-vanishing density and temperature. In section III, we specialize to the case $\mu = T = 0$, and derive an explicit form for the effective potential. We renormalize the coupling constants and derive thereby the renormalized effective potential and gap equations. Section IV is devoted to an analysis of the gap equation and we derive the conditions under which one or the other condensate dominates. Section V contains some conclusions.

II. GENERAL CONSIDERATIONS

We begin with the Lagrangian:

$$\begin{aligned}\mathcal{L} = \bar{\psi}^{(i)} i \not{\partial} \psi^{(i)} + \frac{1}{2} g^2 [\bar{\psi}^{(i)} \psi^{(i)}][\bar{\psi}^{(j)} \psi^{(j)}] \\ + 2G^2 (\bar{\psi}^{(i)} \gamma_5 \psi^{(j)})(\bar{\psi}^{(i)} \gamma_5 \psi^{(j)}) - \mu \psi^{\dagger(i)} \psi^{(i)}. \end{aligned} \tag{2.1}$$

The flavor indices, summed on from 1 to N, have been explicitly indicated. The first quartic term is the usual Gross-Neveu interaction, whereas the second such term, which differs in the arrangement of its flavor indices, induces the pairing force to leading order in $\frac{1}{N}$. In the final term, μ is the chemical potential.

Strictly speaking, a $\langle \psi\psi \rangle$ condensate cannot form, because it breaks the $U(1)$ of Fermion number and hence violates Coleman's theorem [6]. Similarly, $\langle \bar{\psi}\psi \rangle$ as well as $\langle \psi\psi \rangle$ condensates cannot exist at finite temperature in one spatial dimension because of the Mermin-Wagner theorem [6]. Nevertheless, it is meaningful to study the formation of such condensates to leading order in $\frac{1}{N}$, as explained in ref. [7].

Our conventions are: $\gamma^0 = \sigma_1$; $\gamma^1 = -i\sigma_2$; $\gamma_5 = \sigma_3$. The pairing term, proportional to G^2, may then be rewritten:

$$2G^2 \bar{\psi}^{(i)} \gamma_5 \psi^{(j)} \bar{\psi}^{(i)} \gamma_5 \psi^{(j)} = -G^2 [\epsilon_{\alpha\beta} \psi_\alpha^{\dagger(i)} \psi_\beta^{\dagger(i)}][\epsilon_{\gamma\delta} \psi_\gamma^{(j)} \psi_\delta^{(j)}] \ . \tag{2.2}$$

Following standard techniques [8] we add the following terms involving auxiliary fields m, $B^\dagger$, and B:

$$\triangle \mathcal{L} = -\frac{1}{2g^2}[m + g^2 \bar{\psi}\psi]^2 - \frac{1}{G^2}(B^\dagger - G^2 \epsilon_{\alpha\beta} \psi_\alpha^{\dagger(i)} \psi_\beta^{\dagger(i)})(B + G^2 \epsilon_{\gamma\delta} \psi_\gamma^{(j)} \psi_\delta^{(j)}) \ . \tag{2.3}$$

This addition to $\mathcal{L}$ will not affect the dynamics. In $\mathcal{L}' = \mathcal{L} + \triangle\mathcal{L}$, the terms quartic in fermion fields cancel, and we have

$$\mathcal{L}' = \bar{\psi}(i \not\nabla - m - \mu\gamma^0)\psi - \frac{m^2}{2g^2} - \frac{B^\dagger B}{G^2} + B\epsilon_{\alpha\beta}\psi_\alpha^{\dagger(i)}\psi_\beta^{\dagger(i)} - B^\dagger\epsilon_{\alpha\beta}\psi_\alpha^{(i)}\psi_\beta^{(i)} \ . \tag{2.4}$$

We integrate out ψ and $\psi^\dagger$ to obtain the effective action depending on the auxiliary fields m, B and $B^\dagger$:

$$\Gamma_{eff}(m, B, B^\dagger) = \int d^4x(-\frac{m^2}{2g^2} - \frac{B^\dagger B}{G^2}) - \frac{i}{2}Tr\log A^T A - \frac{i}{2}Tr\log[\mathbf{1} + M^2(A^T)^{-1}\sigma_2 A^{-1}\sigma_2] \tag{2.5}$$

where we have subtracted a constant (independent of the auxiliary fields) and have defined:

$$A = \gamma^0(i \not\nabla - m - \mu\gamma^0) = i\partial_0 + i\sigma_3\partial_x - \mu - m\sigma_1 \tag{2.6}$$

so that $A^T = -i\partial_0 - i\sigma_3\partial_x - \mu - m\sigma_1$.

Since we are looking for a vacuum solution, we have assumed in (2.5) that $B, B^\dagger$ and m are constants and have set $M^2 = 4B^\dagger B$. The trace on flavor indices will give a factor N. The large-N limit is achieved by setting $g^2N = \lambda$, and $G^2N = \kappa/4$, and letting $N \to \infty$ with λ and κ fixed. We define the effective potential V_{eff} via

$$\Gamma_{eff} = -N(\int d^2x)V_{eff} \tag{2.7}$$

and we therefore have

$$V_{eff}(m, M) = \frac{m^2}{2\lambda} + \frac{M^2}{\kappa} + V_{eff}^{(1)}(m, M) \ , \tag{2.8}$$

with $V_{eff}^{(1)}(m, M) = \frac{i}{2}[tr\log(A^T A)_{xx} + tr\log(\mathbf{1} + M^2(A^T)^{-1}\sigma_2 A^{-1}\sigma_2)_{xx}]$, where now the trace is only over the spinor indices.

We next generate the local extrema of V_{eff} by solving

$$\frac{\partial V_{eff}}{\partial m^2} = \frac{\partial V_{eff}}{\partial M^2} = 0 \ . \tag{2.9}$$

We evaluate the matrix products in $V_{eff}^{(1)}$ in momentum space, with $\partial_\mu \to ik_\mu$. The traces can be done with the help of

$$tr[\frac{1}{V_0 + \vec{V}\cdot\vec{\sigma}}] = \frac{2V_0}{V_0^2 - \vec{V}^2} \tag{2.10}$$

for any $V_0, \vec{V}$. After some manipulation, equations (2.9) become

$$\frac{1}{2\lambda} = -\frac{\partial V_{eff}^{(1)}}{\partial m^2} = i\int \frac{d^2k}{(2\pi)^2}\frac{[k_0^2 - k_1^2 + \mu^2 + M^2 - m^2]}{D} \tag{2.11}$$

$$\frac{1}{\kappa} = -\frac{\partial V_{eff}^{(1)}}{\partial M^2} = i\int \frac{d^2k}{(2\pi)^2}\frac{[k_0^2 - k_1^2 - \mu^2 - M^2 + m^2]}{D} \tag{2.12}$$

where $D = [k_0^2 - k_1^2 - M^2 + m^2 - \mu^2]^2 - 4[m^2k_0^2 + \mu^2k_1^2 - m^2k_1^2]$. In this expression, k_0 is shorthand for $k_0 + i\epsilon sgnk_0$, where $\epsilon \to 0^+$. This prescription correctly implements the role of μ as the chemical potential.

The equations can be reduced further by doing the k_0 integral. Let us define $k_\pm = \sqrt{b_1 \pm 2b_2}$, where $b_1 = M^2 + m^2 + \mu^2 + k_1^2$, and $b_2 = [M^2m^2 + \mu^2(k_1^2 + m^2)]^{\frac{1}{2}}$. Then evaluating the k_0 integral by contour methods, taking proper account of the $i\epsilon$ prescription mentioned above, we find

$$\frac{1}{2\lambda} = \frac{1}{8\pi}\int_{-\Lambda}^{\Lambda} dk_1[\frac{1}{k_+} + \frac{1}{k_-} + \frac{(M^2+\mu^2)}{\sqrt{M^2m^2 + \mu^2(k_1^2+m^2)}}(\frac{1}{k_+} - \frac{1}{k_-})] \tag{2.13}$$

and

$$\frac{1}{\kappa} = \frac{1}{8\pi}\int_{-\Lambda}^{\Lambda} dk_1[\frac{1}{k_+} + \frac{1}{k_-} + \frac{m^2}{\sqrt{M^2m^2 + \mu^2(k_1^2+m^2)}}(\frac{1}{k_+} - \frac{1}{k_-})] \ . \tag{2.14}$$

The k_1 integrals are logarithmically divergent and we have regularized them by imposing a cutoff Λ. This will be absorbed in the renormalization process to be described in the next section. Note, however, that the combination $\frac{1}{2\lambda} - \frac{1}{\kappa}$ is given by a convergent integral. This fact will ultimately lead to the renormalization-scale independent constant mentioned in the introduction.

We observe from the form of equations (2.11) and (2.12) that the function $V_{eff}^{(1)}$ can be reconstructed by integrating with respect to m^2 and M^2 in the expressions for $\frac{1}{2\lambda}$ and $\frac{1}{\kappa}$. This will determine $V_{eff}^{(1)}$ up to a single constant $V_{eff}^{(1)}(0,0)$, which can be chosen arbitrarily without affecting any physical quantity. Explicitly performing this integration we obtain for the unrenormalized determinant correction to the effective potential

$$V^{(1)}(m,M) = -\frac{1}{2\pi}\int_0^{\Lambda} dk_1[k_+ + k_-] \tag{2.15}$$

To generalize this discussion to the case of non-zero temperature, one returns to eqns. (2.11) and (2.12), and one continues to Euclidean space via the replacement $k_0 \to -ik_4$ with k_4 now considered real. The statistical-mechanical partition function is obtained from the

Euclidean zero temperature path integral by integrating over a finite regime in imaginary time $\tau = it$ from 0 to $\beta = \frac{1}{kT}$. Because of the cyclic property of the trace, the Fermion Green's functions are anti-periodic in τ and one has the replacement

$$\int dk_4 \to \frac{2\pi}{\beta} \sum_n \tag{2.16}$$

where the antiperiodicity gives the Matsubara frequencies:

$$\omega_n = k_{4_n} = \frac{(2n+1)\pi}{\beta} \tag{2.17}$$

To do the sum over the Matsubara frequencies, one uses the calculus of residues to obtain the identity:

$$\frac{2}{\beta} \sum_n f(i\omega_n) = -\sum_s \tanh \frac{\beta z_s}{2} \;\; Res f(z_s) \tag{2.18}$$

where z_s are the poles of $f(z)$ in z in the complex plane; $Res f(z_s)$ is the residue of $f(z)$ at z_s and we have assumed the function $f(z)$ falls off at least as fast as $1/|z|^{1+\epsilon}$ for large $\mid z \mid$. It will be convenient to use:

$$\tanh \frac{\beta z_s}{2} = 1 - 2n_f(z_s)$$

where

$$n_f(z) = \frac{1}{e^{\beta z} + 1}$$

is the usual Fermi-Dirac distribution function.

Rotating equations (2.11) and (2.12) into Euclidean space as described above, we get:

$$\frac{1}{2\lambda} = -\frac{\partial V_{eff}^{(1)}}{\partial m^2} = \int \frac{d^2k}{(2\pi)^2} \frac{[-k_4^2 - k_1^2 + \mu^2 + M^2 - m^2]}{D}$$

$$\frac{1}{\kappa} = -\frac{\partial V_{eff}^{(1)}}{\partial M^2} = \int \frac{d^2k}{(2\pi)^2} \frac{[-k_4^2 - k_1^2 - \mu^2 - M^2 + m^2]}{D}$$

where now

$$\int dk_4 \equiv \frac{2\pi}{\beta} \sum_n \tag{2.19}$$

where $D = [-k_4^2 - k_1^2 - M^2 + m^2 - \mu^2]^2 - 4[-m^2 k_4^2 + \mu k_1^2 - m^2 k_1^2]$. There is no longer any need for an $i\epsilon$ in the definition of k_4. Performing the sums over the Matsubara frequencies

we obtain the unrenormalized form of the equations which are given by the same expression as the zero temperature ones found earlier, with the replacements:

$$\frac{1}{k_+} \to \frac{1}{k_+}(1-2n_f(k_+))$$
$$\frac{1}{k_-} \to \frac{1}{k_-}(1-2n_f(k_-)) \tag{2.20}$$

As before we can integrate this to get the determinant correction to the effective potential which in unrenormalized form is:

$$V^{(1)}(m,M) = -\frac{1}{2\pi}\int_0^\Lambda dk_1[k_+ + k_- + \frac{2}{\beta}\log(1+e^{-\beta k_+}) + \frac{2}{\beta}\log(1+e^{-\beta k_-})] \tag{2.21}$$

III. THE CASE $\mu = T = 0$

Renormalization of the effective potential is best discussed in the context of the zero temperature and density sector of the theory where we can define the renormalized coupling constant in terms of the physical scattering of Fermions at a particular momentum scale. This vacuum sector is interesting in its own right and we shall be able, by analytic means, to derive the result that depending on the relative strengths of the 2 couplings the theory will be in one or another broken phase but never in a mixed phase. Setting $\mu = T = 0$ we obtain

$$\frac{\partial V^{(1)}_{eff}}{\partial m^2} = -\frac{1}{4\pi}\int_0^\Lambda dk_1[(1+\frac{M}{m})\frac{1}{\sqrt{k_1^2+(m+M)^2}} + (1-\frac{M}{m})\frac{1}{\sqrt{k_1^2+(m-M)^2}}] \tag{3.1}$$

$$\frac{\partial V^{(1)}_{eff}}{\partial M^2} = -\frac{1}{4\pi}\int_0^\Lambda dk_1[(1+\frac{m}{M})\frac{1}{\sqrt{k_1^2+(m+M)^2}} + (1-\frac{m}{M})\frac{1}{\sqrt{k_1^2+(m-M)^2}}] \tag{3.2}$$

which is solved by

$$V^{(1)}(m,M) = -\frac{1}{2\pi}\int_0^\Lambda dk_1[\sqrt{k_1^2+(M+m)^2} + \sqrt{k_1^2+(M-m)^2} - 2k_1] \ . \tag{3.3}$$

This can be integrated to give the unrenormalized effective potential:

$$V_{eff}(m,M) = M^2[\frac{1}{\kappa} - \frac{1}{4\pi}] + m^2[\frac{1}{2\lambda} - \frac{1}{4\pi}]$$
$$- \frac{1}{4\pi}[(M+m)^2 ln(\frac{2\Lambda}{M+m}) + (M-m)^2 ln(\frac{2\Lambda}{\mid M-m \mid})] \ . \tag{3.4}$$

We renormalize by demanding that the renormalized couplings κ_R and λ_R satisfy

$$\frac{\partial^2 V_{eff}}{\partial B \partial B^\dagger} \mid_{\substack{M=M_0 \\ m=m_0}} = \frac{4}{\kappa_R} \tag{3.5}$$

and

$$\frac{\partial^2 V_{eff}}{\partial m^2} \mid_{\substack{M=M_0 \\ m=m_0}} = \frac{1}{\lambda_R} \ . \tag{3.6}$$

Here $M = M_0$, $m = m_0$ designates an arbitrary renormalization point on which the couplings will depend. Using these conditions to solve for λ and κ in terms of λ_R and κ_R yields the renormalized form of the effective potential:

$$\begin{aligned} V_{eff} = m^2[a + \frac{1}{4\pi} ln \mid \frac{M^2 - m^2}{\gamma_0} \mid] + M^2[b + \frac{1}{4\pi} ln \mid \frac{M^2 - m^2}{\gamma_0} \mid] \\ + \frac{1}{2\pi} mM ln \mid \frac{M+m}{M-m} \mid \end{aligned} \tag{3.7}$$

where a and b are the following constants:

$$a = \frac{1}{2\lambda_R} - \frac{3}{4\pi}$$

$$b = \frac{1}{\kappa_R} - \frac{1}{2\pi} + \frac{1}{8\pi} \frac{m_0}{M_0} ln \mid \frac{M_0 - m_0}{M_0 + m_0} \mid \tag{3.8}$$

and $\gamma_0 = \mid M_0^2 - m_0^2 \mid$.

Note that the renormalization we have just performed at $\mu = T = 0$ is also sufficient to remove all divergences from the effective potential in the more general case of non-vanishing chemical potential and temperature. The addition of μ and T will only result in finite corrections to the gap equations and therefore to the vacuum values of m and M. We shall return to this point in section V.

The gap equations are properly derived by differentiating V_{eff} with respect to B and m and then setting these derivatives to zero. Because V_{eff} depends only on $B^\dagger B$ and m^2, it will always be possible to have solutions with one of m or B or perhaps both set to zero. The gap equations can be written

$$m[2a + \frac{1}{2\pi} + \frac{1}{2\pi} ln \frac{\mid M^2 - m^2 \mid}{\gamma_0}] - \frac{M}{2\pi} \ ln \mid \frac{M-m}{M+m} \mid = 0 \tag{3.9}$$

and

$$M[b + \frac{1}{4\pi} + \frac{1}{4\pi} ln \frac{\mid M^2 - m^2 \mid}{\gamma_0}] - \frac{m}{4\pi} ln \mid \frac{M-m}{M+m} \mid = 0 \ . \tag{3.10}$$

The solutions $m = m^*$ and $M = M^*$ will give us the local extrema of V_{eff}. The first of these equations is an identity if $m = 0$, and the second if $M = 0$. Also, the values of m and M that solve these equations are physical parameters that must be independent of the renormalization scale γ_0. Thus these equations tell us how a and b individually run with γ_0. We note, however, that if we solve for the combination

$$\delta = a - b = \frac{1}{4\pi}[\frac{M^*}{m^*} - \frac{m^*}{M^*}]ln \mid \frac{M^* - m^*}{M^* + m^*} \mid \tag{3.11}$$

the scale γ_0 drops out. Therefore δ is a true physical parameter in the theory; we shall see in the next section that its value controls which of the two condensates m and M can exist.

IV. ANALYSIS OF THE GAP EQUATIONS

It will be useful in the following to note that, at a solution of the gap equations (3.9) and (3.10), the effective potential takes the simple form

$$V_{eff}(m, M) = -\frac{1}{4\pi}(m^2 + M^2) \ . \tag{4.1}$$

Our goal is to analyze all the solutions of the gap equations and to find the one that produces the global minimum of V_{eff}. This will then represent the true vacuum of the theory.

There are four types of solution to (3.9) and (3.10). The first is simply to set $m = M = 0$, leading of course to $V = 0$. Clearly, from (4.1) we see that if any other solution exists, $V = 0$ cannot be the minimum of V. The second and third types are obtained by setting $M = 0$, $m \neq 0$ and $m = 0$, $M \neq 0$ respectively. If $M = 0$, then from (3.9), we have

$$m^2 = \gamma_0 \ exp[-(1 + 4\pi a)] \tag{4.2}$$

so

$$V_0(m, M = 0) = -\frac{\gamma_0}{4\pi} e^{-(1+4\pi a)} \tag{4.3}$$

(we shall use V_0 to denote values of V_{eff} at solutions of the gap eqn.). Likewise, if $m = 0, M \neq 0$, then from (3.10)

$$M^2 = \gamma_0 \ exp[-(1 + 4\pi b)] \tag{4.4}$$

$$V_0(m = 0, M) = -\frac{\gamma_0}{4\pi} \ e^{-(1+4\pi b)} \ . \tag{4.5}$$

Thus we see that

$$V_0(m=0,M) < V_0(m,M=0) \quad \text{if } \delta > 0 \tag{4.6}$$

and

$$V_0(m,M=0) < V_0(m=0,M) \quad \text{if } \delta < 0\ . \tag{4.7}$$

The fourth case is when both m and M are non-vanishing. It is then convenient to define $\rho = \frac{M}{m}$ and to combine the gap equations in the form

$$\delta = (\rho^2-1)[b + \frac{1}{4\pi} + \frac{1}{4\pi} ln(\frac{m^2 \mid \rho^2-1 \mid}{\gamma_0})] \tag{4.8}$$

and

$$\delta = \frac{1}{4\pi}\frac{(\rho^2-1)}{\rho}\ ln\ \mid \frac{\rho-1}{\rho+1} \mid\ . \tag{4.9}$$

Both these equations are even in ρ, so we may take $\rho > 0$ for convenience. Eqn. (4.9) tells us immediately that if $\delta > 0$, $0 < \rho < 1$, and if $\delta < 0$, $\rho > 1$. Furthermore, the r.h.s. of (4.9) is bounded between $-\frac{1}{2\pi}$ and $\frac{1}{2\pi}$. Hence we conclude: If $\mid \delta \mid > \frac{1}{2\pi}$ there is no solution with both m and M non-vanishing. If $\mid \delta \mid < \frac{1}{2\pi}$, there is such a solution, with the property that $m > M$ if $\delta > 0$ and $M < m$ if $\delta < 0$.

It remains to decide whether $V_0(m,M)$ can be the global minimum. To this end, it is convenient to reexpress the gap equations once more in the following form:

$$-(1+4\pi a) = ln\ \frac{m^2 \mid \rho^2-1 \mid}{\gamma_0} - ln\{\ \mid \frac{\rho-1}{\rho+1} \mid^{\rho}\} \tag{4.10}$$

and

$$-(1+4\pi b) = ln\ \frac{m^2 \mid \rho^2-1 \mid}{\gamma_0} - ln\{\ \mid \frac{\rho-1}{\rho+1} \mid^{\frac{1}{\rho}}\}\ . \tag{4.11}$$

From these, making use of eqs. (4.1), (4.3) and (4.5), we immediately obtain

$$V_0(m=0,M) = -\frac{\gamma_0}{4\pi}\ e^{-(1+4\pi b)} = g_1(\rho)V_0(m,M) \tag{4.12}$$

and

$$V_0(m,M=0) = -\frac{\gamma_0}{4\pi}\ e^{-(1+4\pi a)} = g_2(\rho)V_0(m,M) \tag{4.13}$$

where

$$g_1(\rho) = \frac{(1+\rho)^{1+\frac{1}{\rho}} \mid 1-\rho \mid^{1-\frac{1}{\rho}}}{1+\rho^2} \tag{4.14}$$

and

$$g_2(\rho) = \frac{(\rho+1)^{\rho+1} \mid \rho-1 \mid^{1-\rho}}{1+\rho^2} \ . \tag{4.15}$$

Eq. (4.12) is the relevant comparison if $\frac{1}{2\pi} > \delta > 0$ and $0 < \rho < 1$, whereas eq. (4.13) is relevant for $0 > \delta > -\frac{1}{2\pi}$ and $\rho > 1$.

We observe, however, that $g_2(\frac{1}{\rho}) = g_1(\rho)$, so both cases reduce to the following: if we can show that $g_1(\rho) > 1$ in the range $0 < \rho < 1$, then $V_0(m, M)$ is never the global minimum (recall that the $V_0's$ are all < 0). On the other hand, if $g_1(\rho) < 1$ in this range, it will be possible to have $V_0(m, M)$ be the global minimum.

To settle this question, write $g_1 = e^h$, with

$$\begin{aligned} h(\rho) &= (1+\frac{1}{\rho})\ ln\ (1+\rho) + (1-\frac{1}{\rho})\ ln\ (1-\rho) - \ ln\ (1+\rho^2) \\ &= ln\ [\frac{1+\rho}{1+\rho^2}] + \frac{1}{\rho}\ ln\ (1+\rho) + (1-\frac{1}{\rho})\ ln\ (1-\rho)\ . \end{aligned} \tag{4.16}$$

In the range of interest, $\rho^2 < \rho$, so the r.h.s. is a sum of positive terms. Hence $h(\rho) > 0$ and $g_1(\rho) > 1$.

We conclude that the global minimum of V_{eff} has $M = 0, m \neq 0$ if $\delta < 0$, and $m = 0, M \neq 0$ if $\delta > 0$.

V. CONCLUSIONS

From eqn. (2.21) we can see that the corrections due to non-vanishing temperature and density do not affect the ultraviolet behavior of the integrand in the k_1 integral defining $V^{(1)}$. Therefore, the renormalization that we have performed at $\mu = T = 0$ in section III suffices to remove the ultraviolet divergences from the effective potential, and will allow us to send the cutoff to infinity. It is perhaps worth recording the complete result explicitly. We find, from eqns. (3.5) and (3.6), that

$$\frac{1}{2\lambda} = a + \frac{1}{4\pi} + X \tag{5.1}$$

$$\frac{1}{\kappa} = b + \frac{1}{4\pi} + X \tag{5.2}$$

where a and b are defined by eqn. (3.8), and X is a divergent integral given by

$$X = \frac{1}{4\pi}\int_0^\Lambda dk_1[\frac{1}{\sqrt{k_1^2+(m_0+M_0)^2}} + \frac{1}{\sqrt{k_1^2+(m_0-M_0)^2}}]$$
$$= \frac{1}{2\pi}\ [log\ (\frac{2\Lambda}{\sqrt{\gamma_0}})] + \text{ terms which vanish as } \Lambda\to\infty\ . \tag{5.3}$$

Thus the full renormalized effective potential may be written

$$V_{eff} = \alpha_1 m^2 + \alpha_2 M^2 - \frac{1}{2\pi}\int_0^\infty dk_1[k_+ + k_- + \frac{2}{\beta}log\ (1+e^{-\beta k_+})$$
$$+ \frac{2}{\beta}log\ (1+e^{-\beta k_-})$$
$$- 2k_1 - (\frac{m^2+M^2}{2})(\frac{1}{\sqrt{k_1^2+(m_0+M_0)^2}} + \frac{1}{\sqrt{k_1^2+(m_0-M_0)^2}})]\ . \tag{5.4}$$

where $\alpha_1 = \frac{1}{4\pi}(1+4\pi a)$ and $\alpha_2 = \frac{1}{4\pi}(1+4\pi b)$. If $\alpha_1 < \alpha_2$, then at $\mu = T = 0$ the vacuum has $m^2 = m_F^2 \equiv \gamma_0 e^{-4\pi\alpha_1}$ and $M^2 = 0$. Here m_F is the dynamically generated fermion mass. It is convenient to choose the renormalization scale so that $m_F^2 = \gamma_0$. This entails $\alpha_1 = 0, \alpha_2 > 0$. Furthermore, we are free to choose $M_0 = 0$, so that $m_0 = m_F$. Then V_{eff} takes the form

$$V_{eff} = \alpha_2 M^2 - \frac{1}{2\pi}\int_0^\infty dk_1[k_+ + k_- + \frac{2}{\beta}(log\ (1+e^{-\beta k_+}) + log\ (1+e^{-\beta k_-})) - 2k_1$$
$$- (m^2+M^2)\frac{1}{\sqrt{k_1^2+m_F^2}}]\ . \tag{5.5}$$

We observe that if we set $M = 0$ in this expression, we obtain, with $E = \sqrt{k^2+m^2}$,

$$V_{eff}(m^2,T,\mu) = \frac{m^2}{4\pi}[log\ \frac{m^2}{m_F^2} - 1]$$
$$- \frac{2}{\beta}\int_0^\infty \frac{dk}{2\pi}\ log\ [(1+e^{-\beta(E+\mu)})(1+e^{-\beta(E-\mu)})] \tag{5.6}$$

which is the effective potential for the Gross-Neveu model in agreement with refs. [9] and [10], and furthermore at $T = 0$ the integral can be done to give an explicit form that agrees with the results of ref. [11].

Similarly, in the opposite case $\alpha_2 < \alpha_1$, we have, in the $\mu = T = 0$ vacuum, $m^2 = 0$ and $M^2 = \Delta^2 \equiv \gamma_0 e^{-4\pi\alpha_2}$, where Δ is the dynamically generated gap. So we choose $\alpha_2 = 0, \alpha_1 > 0$, and $m_0 = 0$, $\Delta^2 = \gamma_0 = M_0^2$. The effective potential becomes

$$V_{eff} = \alpha_1 m^2 - \frac{1}{2\pi}\int_0^\infty dk_1[k_+ + k_- + \frac{2}{\beta}(log\ (1+e^{-\beta k_+}) + log\ (1+e^{-\beta k_-})) - 2k_1$$
$$- (m^2+M^2)\frac{1}{\sqrt{k_1^2+\Delta^2}}] \ . \tag{5.7}$$

When $m^2 = 0$, this expression gives us the effective potential at finite temperature for the pure Cooper-pairing model considered in ref. 1. Explicitly we have

$$V_{eff} = \frac{M^2}{4\pi}[ln\frac{M^2}{\Delta^2} - 1] - \frac{2}{\beta}\int_0^\infty \frac{dk}{\pi} log\ [1 + e^{-\beta\sqrt{k^2+M^2}}] \ . \tag{5.8}$$

Note that it is independent of the chemical potential, as was the case at $T = 0$. For $T >> M$ we can expand the integral in eq. (5.8) to obtain [12]

$$V_{eff} = \frac{M^2}{2\pi}[log\ (\frac{\pi T}{\Delta}) - \gamma] \ , \tag{5.9}$$

where γ is Euler's constant. The minimum of this function occurs at $M = 0$, which means that the condensate vanishes for large T, as expected. This feature is also borne out by numerical evaluation of eq. (5.8) [13].

In this paper, we have derived the general forms for the effective potential in leading order in $\frac{1}{N}$, eqs. (5.4)-(5.6). We have analyzed the case $\mu = T = 0$ in detail, showing how the phase structure is governed by the relative magnitude of the two constants α_1 and α_2. For $\mu = T = 0$ this structure is remarkably symmetric in the two condensates m and M. We expect this symmetry to disappear in the general case, however, because μ acts to suppress m. To study this will require careful numerical analysis of the integrals in eqs. (5.5)and(5.7), in order to see how the extrema of V_{eff} depend on the parameters α_1, α_2, μ and T. This work is currently being actively pursued. It is our hope that the results may be useful for a variety of condensed matter systems, for QCD, and perhaps may have cosmological implications as well.

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